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Nonlinear Fokker–Planck–Landau integral propagator (II): transport far from equilibrium

J M Donoso[1] **and J J Salgado**[2]

[1] Departamento de Física Aplicada, ETSI Aeronáuticos, Universidad Politécnica de Madrid, 28040 Madrid, Spain
[2] European Space Astronomy Centre, ESA, Ap.50727, 28080 Madrid, Spain

E-mail: josemanuel.donoso@upm.es and jesus.salgado@sciops.esa.int

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Abstract

We present a time evolving path-integral method for solving the Landau–Fokker–Planck equation to compute kinetic transport coefficients in a fully ionized plasma. The electron distribution function is advanced in time by means of the conservative short-time propagators, which we previously obtained. The validated integral operator takes into account both electron–electron and electron–ion collisions without linearizing the original Fokker–Planck collisional operator. The resulting integral formulation in velocity space is applied here to evaluate the local transport coefficients if inhomogeneities in configuration space appear. We define an effective source term through a flux particle balance in a thin slab of plasma, which leads to a nonhomogeneous Fokker–Planck equation. Hence, this new term locally models the so-called Vlasov term appearing in the general kinetic equation. Arbitrary departures from Maxwellian equilibrium can be dealt with this effective source term that preserves the positiveness of the electron distribution function, even in the runaway limit. For small perturbations of the equilibrium, the classical Spitzer and Harm transport coefficients are recovered, while a very strong reduction of the heat flux takes place for large temperature gradients, as predicted by some authors in different theories.

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1. Introduction

The Fokker–Planck equation plays a fundamental role in the description of collisional plasmas ranging over a wide class of realistic scenarios, from astrophysics to inertial confinement systems, see, for instance, the exhaustive and amenable review [1] . Moreover, this equation also appears in many branches of physics in relation to a wide class of phenomena [2, 3].

Many attempts have been devoted to properly study the nature of solutions for this nonlinear integro-differential equation in plasmas, as well as to solve it, using analytical approaches or numerical methods.

The semi-analytical solutions are always obtained under fair approximations of the original Fokker–Planck equation (FPE), in most cases after having proceeded by the linearization of the full nonlinear operator. Further simplifications are related to some scale properties of the system under inspection, in order to get a more manageable collisional term.

On the other hand, the design of specialized numerical methods on solving both, full and simplified Fokker–Planck equations, has attracted great attention in the last two decades. The matter of constructing feasible numerical methods is still a subject of continuous investigation because any numerical algorithm should be coherent with the main characteristic properties of the collisional operator, such as conservation properties and the kinetic entropy production. Conservative and entropic numerical schemes [4–7] are important in order to compute macroscopic properties of the plasma fluid, such as the so-called transport coefficients [8–10]. In most cases, numerical computation is required, even if the electron distribution function f_e is approximated up to first order in any kind of expansion and the collision operator is linearized. One of the most useful linearization procedures, apart from the drastic simplification introduced in the Bhatnagar–Gross–Krook operator [11, 12], is the well-known Chapman–Enskog expansion [13, 14], which was applied by Spitzer and Harm to perform one of the most discussed evaluations of typical transport coefficients [8].

Another widely extended method consists in using Laguerre polynomial or cartesian tensor expansions for f_e, as in the calculation done by Epperlein and Haines in [15]. In all cases, the collision operator is linearized and the distribution function expansion is cut off after a few number of terms, even for new interesting numerical approaches as that proposed in [16]. This series truncation may lead to the unphysical description of f_e, in the sense that the electron population is not well described in the range of high velocities. Consequently, in order to find a feasible description of both slow and fast electrons' populations, it would be desirable to deal with the full nonlinear nature of the collision operator. This is an important task in numerical computation, since fast electrons have long mean free path exceeding the characteristic length of temperature and density gradients, modifying heat and current flows. In this sense, a low density of suprathermal or runaway electrons in presence of an electric field can dominate transport, which becomes non-local, and it may be responsible for some other important effects, as magnetic field generation inside the plasma [17].

In a previous paper [18], we gave a procedure to numerically solve the nonlinear FPE in a fully ionized collisional plasma, thanks to the construction of a suitable short-time integral propagator. This propagator plays the role of an effective integral collision operator in velocity space, enabling a way to implement in time the distribution function through a path-integral formalism [2, 19–22]. In a different frame, the path-integral method in statistical physics had already been related, among others, to plasma theory by Dagan and Hortwitz in [23] and, previously by Chandrasekhar [24], being a pioneer in dealing with the dynamics of particles under deterministic and stochastic forces. We, firstly, applied the path-sum method to the plasma FPE in our earliest work [25]. More recently, other authors, as Bizarro *et al* in [26] have applied the path-sum approach to the same problem, but using the common Gaussian propagator. Although our improved propagator was derived for the homogeneous equation, assuming a spatially independent distribution function, this integral method can be easily extended to solve the nonhomogeneous equation when spatial dependence or source terms

appear under a more general formulation of the FPE. We remark that the nonunique form of the propagator [20, 21] enables us to recalculate it in such a way that all conservative properties of the integro-differential operator are preserved. In this sense, all the information to describe the interactions between charged particles is obtained from the collisional integro-differential equation, using the drift and diffusion coefficients defined for it. Hence, no linearization of the collisional operator is used, in the sense that, at any time step, the propagator is recomputed using the full distribution function, from which we get the new coefficients. Roughly speaking, the new integral collision operator works in the short-time regime as an exact one, since the diffusion coefficients in a fully ionized plasma vary slowly in time.

As it is well known, the main task in kinetic plasma theory is to provide correct values for the so-called transport coefficients appearing in the fluid description of plasmas. Thus, as an application of our numerical integral method, we reproduced in [18] the values of the classical transport coefficients derived by Spitzer and Harm in [8] for very small (electric and temperature gradient) perturbative fields. However, even with the same equation used in the Spitzer–Harm problem, except, of course, for the collisional term, deviations from classical values appear when the perturbation from equilibrium begins to grow. The meaning of these deviations was analysed and justified by the use made of a formal nonhomogeneous term, termed in [18] as ρ_{spt}, taken from the classical calculations, although we used the full nonlinearized collisional operator. We have found that the negative values in the tails of f_e are very similar to those in the classical Spitzer–Harm calculation. Therefore, nothing significantly new came out of our previous analysis, except, of course, the practical validation of a very different time evolving method to compute the local transport coefficients. It, therefore makes little sense to use this term in a full nonlinear formulation of the problem, that has to be undertaken in a new approach as that we are giving here.

Our main aim in the present paper is to go beyond the limits imposed by the linearization method. The strongest argument to abandon the frame in which the results of [18] were obtained is, thus, the merging of negative amplitudes for large values of $|\mathbf{v}|$ and the electric field strength $|\mathbf{E}|$. The reason for the failure of such a basic principle as the positiveness of the distribution function, is no other than the polynomial approximations contained in the nonhomogeneous term as a function of the temperature gradient ∇T and E, as well as the use of the truncated expansion of f_e in [8]. Moreover, as we shall see, if $\mathbf{E}$ is omitted from ρ_{spt} and it is included in the advance scheme as a convective term, the same nonphysical behaviour in f_e is observed. Then, if we intend to develop a realistic fully nonlinear model to evaluate the transport processes, we are forced to abandon the formal source ρ_{spt} (or any similar one coming from linearized schemes) and to construct an alternative physically meaningful term having similar properties. At the same time, however, this term must be constructed using no approximation either for the distribution f_e or for the collisional operator in the Fokker–Planck–Landau form, taking into account that classical Coulomb collisions are dominant in thermal heat flux [27–29]. Yet, working under the perspective of local transport calculus, our effective source term ρ is proposed under a clear physical sense, following the idea pointed out by Soler in [18], through a particle flux balance on the surface of a small volume in configuration space. In this sense, the proposed ρ locally models the so-called Vlasov term of a more general plasma kinetic equations. The new source term provides the same results obtained with ρ_{spt} for very small deviations from the Maxwellian thermodynamic equilibrium, but it allows the analysis of increasing departures from this state. We have found significant flux inhibition for large electric field and/or steep gradients, as predicted by some authors [30–32].

2. Integral collision operator in velocity space

Under a quite general formulation, the nonhomogeneous Fokker–Planck equation in kinetic theory

$$\frac{\partial f}{\partial t} = \mathcal{L}_{FP}(\mathbf{q},t)f(\mathbf{q},t) + \rho(\mathbf{q},t) = -\frac{\partial}{\partial q_i}\left[\mathcal{A}_i(\mathbf{q},t) - \frac{\partial}{\partial q_j}\mathcal{D}_{ij}(\mathbf{q},t)\right] f + \rho(\mathbf{q},t) \tag{1}$$

describes the time evolution of a distribution function $f(\mathbf{q},t)$, depending on time and the six variables $\mathbf{q}$ of the vector $\{\mathbf{r},\mathbf{v}\}$ for a point in the phase space. The term ρ is understood as a source term . The components $\mathcal{A}_i$ of the drift vector $\mathcal{A}$, as well as the diffusion tensor elements $\mathcal{D}_{ij}$ and ρ, may also depend on f, as in the case of the Fokker–Planck equation in plasma physics. In this case, we shall refer to the above equation as a nonlinear FPE as in the interesting book [3].

The above equation also describes the evolution of a propagator or Green function $\Pi(\mathbf{q},t \mid \mathbf{q}',t')$ that satisfies the same equation with an impulsive source $\rho = \delta(\mathbf{q}-\mathbf{q}')\delta(t-t')$ at time t' . If this propagator can be found, the solution for f is given by the time integral evolution as

$$f(\mathbf{q},t) = \int f(\mathbf{q}',t')\Pi(\mathbf{q},t \mid \mathbf{q}',t')\,\mathrm{d}\mathbf{q}' + \int \mathrm{d}\mathbf{q}' \int_{t'}^{t} \rho(\mathbf{q}',\tau)\Pi(\mathbf{q},t \mid \mathbf{q}',\tau)\,\mathrm{d}\tau. \tag{2}$$

This integral equation exhibits a clear probabilistic meaning when Π is understood as a transition probability from a source point $\mathbf{q}'$ at time t' to a field point $\mathbf{q}$ at time $t > t'$, although, in general, Π may not have this probability sense [33]. In most cases, the differential equation for the propagator Π cannot be solved if the same equation for f is not solvable. Fortunately, it is possible sometimes to obtain an approximate nonunique propagator $P_\tau(\mathbf{q},t+\tau;\mathbf{q}',t) \approx \Pi(\mathbf{q},t+\tau;\mathbf{q}',t)$, only valid in the short-time regime of the evolution. Such a short-time propagator can be found by several methods, such as the construction of an auxiliary solvable Fokker–Planck equation, as we suggested in [34]. A well-known short-time propagator for the transition from time $t' = t$ to time $t+\tau$ is the Gaussian distribution [2, 26], which in N-dimensional space reads

$$P_\tau(\mathbf{q},t+\tau;\mathbf{q}',t) = \frac{\exp\left[-\mathcal{D}'^{-1}_{ij}U_iU_j/4\tau\right]}{\|\mathcal{D}'\|^{1/2}(4\pi\tau)^{N/2}} \tag{3}$$

for a nonsingular diffusion tensor, $\|\mathcal{D}\| \neq 0$. Here, we have defined the vector $\mathbf{U} = \mathbf{q} - \mathbf{q}' - \mathcal{A}'\tau$. In (3) primes indicate that the corresponding functions are evaluated in source points (pre-points) $\mathbf{q}'$ at time t, instead of being computed at field points (post-points) $\mathbf{q}$.

Our present work deals with solving the plasma physics Fokker–Planck equation for the evolution of the electron distribution function f_e

$$\frac{\partial f_e}{\partial t} = -\frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{v}f_e - \frac{\partial}{\partial \mathbf{v}}\cdot\frac{\mathbf{F}}{m_e}f_e + C(f_e,f_e), \tag{4}$$

where $\mathbf{F}$ is the external force, $(\partial/\partial\mathbf{r})\cdot\mathbf{v}f_e$ is usually called the Vlasov term, describing spatially inhomogeneities effects on $f_e = f_e(\mathbf{r},\mathbf{v},t)$. In the following, we drop the argument $\mathbf{r}$ in the notation of f_e and other functions derived from it. The integro-differential operator $C = C^{e/e} + C^{e/i}$ is the collision term, that acts on velocity variables and can be written as

$$C(f_e,f_e) = \mathbf{L}_{FP}f_e = -\frac{\partial}{\partial \mathbf{v}}\cdot\left[\mathbf{D}^e(\mathbf{v},t) - \frac{\partial}{\partial \mathbf{v}}\cdot\mathbb{D}^e(\mathbf{v},t)\right] f_e(\mathbf{v},t). \tag{5}$$

The drift vector $\mathbf{D}^e$ and the diffusion tensor $\mathbb{D}^e$ components include themselves the contribution of the electron–electron and electron–ion collisions through the additive relation

$D^e = D^{e/e} + D^{e/i}$. These coefficients can be computed using Trubnikov's Rosenbluth-like (anisotropic) potentials [5, 10]

$$\psi_b(\mathbf{v},t) = -\frac{1}{8\pi}\int |\mathbf{v}-\mathbf{v}'|\, f_b(\mathbf{v}',t)\,\mathrm{d}\mathbf{v}', \qquad \varphi_b(\mathbf{v},t) = -\frac{1}{4\pi}\int \frac{f_b(\mathbf{v}',t)}{|\mathbf{v}-\mathbf{v}'|}\,\mathrm{d}\mathbf{v}', \tag{6}$$

for electron test particle (e) in a medium of particles of species b (here $b = e, i$) as

$$D_\alpha^{e/b} = -L^{e/b}\left(1+\frac{m_e}{m_b}\right)\frac{\partial\varphi_b}{\partial v_\alpha}, \qquad D_{\alpha\beta}^{e/b} = -L^{e/b}\frac{\partial^2\psi_b}{\partial v_\alpha \partial v_\beta}. \tag{7}$$

Here, the standard notation [10] is followed by all physical quantities. For the absolute elementary charge e and masses m_b, one has $L^{e/e} = \lambda 16\pi^2 e^4/m_e^2$, $L^{e/i} = Z^2 L^{e/e}$ if Z is the relative ionic charge and λ is the Coulomb logarithm [10], taken as a constant. In this work, we assume the ions at rest in the lab frame and distributed in velocities according to a Dirac δ function, with density $n_i = n_e/Z$ for plasma neutrality, as $f_i = n_i\delta(\mathbf{v})$.

The term $\mathbf{F}$ usually refers to the Lorentz force $-e[\mathbf{E}+\mathbf{v}\times\mathbf{B}]$, which can be directly added to the drift vector in the collisional operator if the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ do not explicitly depend on $\mathbf{v}$. The resulting FPE has the general form given by (1) with a drift vector $\mathcal{A} = \{\mathbf{v}, \mathbf{D}^e + \mathbf{F}/m_e\}$ and a singular symmetrical diffusion tensor with zero components for $\mathcal{D}_{v_i x_j}$ and $\mathcal{D}_{x_i x_j}$. In this sense, one must observe that the short-time propagator (3) is not valid to provide an advancing scheme for the present problem. Otherwise, if it is possible to derive such a suitable propagator, this would operate in a six-dimensional space, making the computational effort to be rather cumbersome, and almost impossible without further simplifications. This is why we propose here a simple model, in order to locally describe the collisional plasma, without inserting explicitly the divergence of flux particle in a spatial point $(\partial/\partial\mathbf{r})\cdot\mathbf{v}f_e$ or the Vlasov term, as we shall indicate in the following sections. We adopt here the advancing scheme based on splitting the collisional operator into two parts, due to electron–electron and electron–ion interactions.

As in [5], we use a cylindrical coordinates representation of the equation, assuming f_e as depending on both axial and radial velocity components $v_\parallel = v_z$ and $v_\perp^2 = v_x^2 + v_y^2$, parallel and perpendicular to an externally applied field. The cylindrical coordinates $(v_\perp, v_\parallel, \phi)$ are defined in terms of the spherical ones (v, θ, ϕ) as usual $v_\perp = v\sin\theta$, and $v_\parallel = v\cos\theta$. In these coordinates a FPE depending only on velocity variables, with an operator in the form of (5), is transformed into the new FPE

$$\frac{\partial f^*}{\partial t} = -\left[\frac{\partial}{\partial v_\perp}\left(D_\perp^* - \frac{\partial}{\partial v_\parallel}D_{\perp\parallel} - \frac{\partial}{\partial v_\perp}D_{\perp\perp}\right) + \frac{\partial}{\partial v_\parallel}\left(D_\parallel - \frac{\partial}{\partial v_\perp}D_{\perp\parallel} - \frac{\partial}{\partial v_\parallel}D_{\parallel\parallel}\right)\right] f^* \tag{8}$$

for the new function f^* defined as $f^*(v_\perp, v_\parallel; t) = 2\pi v_\perp f_e(\mathbf{v};t)$ and $D_\perp^* = D_{\phi\phi}/v_\perp + D_\perp$, where all the coefficients refer to the electron–electron collisions, satisfying (7) for $(\alpha,\beta) = (\perp,\parallel)$ and $D_{\phi\phi} = -L^{e/e}(\partial\psi_e/\partial v_\perp)/v_\perp$. In order to simplify the final form of the short-time propagator, the drift vector components can be redefined in terms of two new effective coefficients $\mathbf{A}$ through the relations

$$\frac{D_{\phi\phi}}{v_\perp} + D_\perp = \frac{D_{\perp\perp}}{v_\perp} + A_\perp \qquad \text{and} \qquad D_\parallel = \frac{D_{\perp\parallel}}{v_\perp} + A_\parallel. \tag{9}$$

Thus, f_e can be implemented in the short-time regime of evolution by means of the approximate Green's function $P_\tau(\mathbf{v},\mathbf{v}'|t)$, given by

$$P_\tau = v_\perp \frac{i_0(2v_\perp(v_\perp - U)/(4D'_{\perp\perp}\tau))}{2\tau\sqrt{4\pi\, D'_{\perp\perp}\tau\, D'_t}}\exp\left[-\frac{D'_{\parallel\parallel}U^2 - 2D'_{\perp\parallel}UV + D'_{\perp\perp}V^2}{4\tau\, D'_t}\right], \tag{10}$$

which was derived and tested in our previous work [18]. Here $U = v_\perp - v'_\perp - A'_\perp\tau$ and $V = v_\parallel - v'_\parallel - A'_\parallel\tau - F'_\parallel\tau/m_e$, where we have added the parallel force $F_\parallel$ (in the following,

$-eE/m_e$) due to an externally applied field. D_t denotes the determinant of the 2×2 nonsingular cylindrical diffusion matrix associated with the formal vector $\{v_\perp, v_\parallel\}$. Primes here indicate that the coefficients have to be computed in the pre-point variables $\mathbf{v}'$ in time t. The continuous bounded function i_0 is related to the zero-order modified Bessel function of the first kind I_0 as $i_0(q) = I_0(q)\exp(-q)$. It is important to realize that (10) is suitable for any operator that can be cast into the form of (8) as, for instance, in dealing with the interaction of plasma and radio-frequency waves [5], relativistic plasmas [35] and generalized collision operators as in partially degenerate plasmas [36]. As said before, without loss of generality, the ions are assumed to be at rest, being the Rosenbluth-like potentials describing the contribution of the ions to the collision operator $\psi_i = -n_i L^{e/i} v/8\pi$ and $\varphi_i = -n_i L^{e/i}/4\pi v$. Otherwise, if the ions move at constant flow velocity $\mathbf{u}_i$, it is sufficient to simply translate the $\mathbf{v}$ argument of f_e into $\mathbf{v} - \mathbf{u}_i$. The resulting operator describing the electron–ion interaction, denoted by $(\partial f/\partial t)_i$ as usual, has the form of (5) with drift and diffusion coefficients given by the relations

$$D_i^{e/i}(\mathbf{v}) = -\left(1 + \frac{m_e}{m_i}\right)\frac{n_i L^{e/i}}{4\pi}\frac{v_i}{v^3} \qquad \text{and} \qquad D_{ij}^{e/i}(\mathbf{v}) = \frac{n_i L^{e/i}}{8\pi}\frac{\delta_{ij}v^2 - v_i v_j}{v^3}, \tag{11}$$

in cartesian coordinates. Explicitly, this contribution is best expressed in spherical coordinates as

$$\left(\frac{\partial f_e}{\partial t}\right)_i = \frac{n_i L^{e/i}}{8\pi v^3}\left[\nabla_\theta^2 + \mu v\frac{\partial}{\partial v}\right] f_e = \frac{n_i L^{e/i}}{8\pi v^3}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \mu v\frac{\partial}{\partial v}\right] f_e \tag{12}$$

for $f_e = f_e(v, \theta, t)$, where we have introduced the mass ratio factor $\mu = 2m_e/m_i$ and the Laplace–Beltrami operator ∇_θ^2 in spherical coordinates for azimuthally independent f. The term proportional to $\mu \neq 0$ plays the role of an almost negligible friction force that only affects very slow electrons. If we assume infinite massive ions $\mu = 0$, the electron–ion operator $\mathbf{L}_{FP}^{e/i}$ will only involve angular variables in velocity space, meaning that it only scatters electron velocity direction. In this case, this differential operator admits a well-known integral propagator $P_\tau^i = \exp\left(\tau\mathbf{L}_{FP}^{e/i}\right)\delta(\mathbf{v} - \mathbf{v}')$ which is valid for any value of time τ. It is given as a series of Legendre Polynomials $P_k(\cos\theta)$ by

$$P_\tau^i = \Pi^{e/i}(\mathbf{v}, t+\tau \mid \mathbf{v}', t) = \frac{\delta(v - v')}{2\pi v^2}\sum_{k=0}^{\infty}\left(k + \frac{1}{2}\right)P_k(\cos\theta)P_k(\cos\theta')\,\mathrm{e}^{-\tau\lambda_k}, \tag{13}$$

with $\lambda_k = ((k + 1/2)^2 - 1/4)n_i L^{e/i}/(8\pi v'^3)$. For a finite mass ratio μ, this expression should be corrected by solving the Cauchy problem associated with (12) for an initial distribution $\delta(\mathbf{v} - \mathbf{v}')$. To do this, it is sufficient to assume the new function P_τ^i in the form $\sum_k g_k(v, \tau)P_k(\cos\theta)$ and solve the resulting first-order differential equation for g_k. Through this procedure we have found that P_τ^i essentially preserves the standard form of (13) since it is given by

$$P_\tau^i = \frac{\delta(v - v'\xi^{1/3})}{2\pi v^2}\sum_{k=0}^{\infty}\left(k + \frac{1}{2}\right)P_k(\cos\theta)P_k(\cos\theta')\,\mathrm{e}^{-\tau\lambda_k\zeta}, \tag{14}$$

where the positive correction factors $\xi = 1 + 3\mu n_i L^{e/i}\tau/(8\pi v'^3)$ and $\zeta = \ln\xi/(\xi - 1)$ satisfy $\xi = \zeta = 1$ in the limit $\mu \to 0$ and (13) is recovered from the most general expression (14). Observe that this probability transition is valid for any value of τ, as well as it coincides with the exact propagator Π in the limit of the so-called Lorentz plasma, i.e. when $Z \to \infty$ and electron–electron collisions are neglected [37]. Other effects, as radio-frequency or inverse bremsstrahlung heating processes in the plasma [28], can be taken into account by including them as effective diffusion or drift coefficients in either (10) or (14).

The computation of the series in (14) may involve a large number of terms for small values of the exponential argument. Then, it would be desirable to accelerate the convergence of the series by using the approximate and very accurate formula we gave in [34] for the following standard series with $\beta > 0$, after a suitable identification of the terms appearing in (14) or (13). If we define

$$P_\beta(\theta) = \mathrm{e}^\beta \sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right) \mathrm{e}^{-\beta(2k+1)^2} P_k(\cos\theta) P_k(\cos\theta') \tag{15}$$

for small β we have

$$P_\beta(\theta) \simeq \frac{i_0(\sin\theta \sin\theta'/2\beta)}{2\beta(\mathrm{e}^{1/\beta} - 1)} \exp\left[\frac{1 + \cos\theta\cos\theta' + \sin\theta\sin\theta'}{2\beta}\right], \tag{16}$$

where i_0 is the same function defined for (3). Although this expression was derived for small values of β, one can verify that in the limit of large β it approaches the correct asymptotic limit for the series. In fact, (16) approximates the exact series within an error lower than 5% for any $\beta > 0$. Moreover, in spite of the fact that we have taken a static δ ion distribution function for f_i, if it is approximated by a Maxwellian it would lead to a very similar representation of the definite integral collisional operator when both species are at comparable temperatures. Under a computational point of view, the static representation of f_i is well behaved and, if another one were chosen, if it could be depicted as a superposition of Dirac δ functions in the electron reference frame, leading to an effective propagator P_τ^i given as a product of a few set of propagators of the same form, each one centred in a given point of the velocity space. For the purposes of this work, the approach given above is enough to compute local transport coefficients in a similar way to that done in the classical calculation.

Finally, if P_τ^e represents the short-time propagator in the form of (10) counting itself electron–electron collisions and external forces, fe can be advanced in time by using as a probability transition $P_\tau = P_\tau(\mathbf{v}, t+\tau; \mathbf{v}', t)$ either

$$P_\tau = \int P_\tau^e(\mathbf{v}, \mathbf{w}) P_\tau^i(\mathbf{w}, \mathbf{v}')\, \mathrm{d}\mathbf{w} \qquad \text{or} \qquad P_\tau = \int P_\tau^i(\mathbf{v}, \mathbf{w}') P_\tau^e(\mathbf{w}', \mathbf{v}')\, \mathrm{d}\mathbf{w}' \tag{17}$$

by means of (2) in the velocity space, with $\rho = 0$ if only collisions and external forces govern the f_e motion. Instead of using this propagator, one can advance f_e with P_τ^i (or P_τ^e) and apply the same integral scheme to the resulting function with P_τ^e (or P_τ^i), as we showed in [34], where the advancing process is thoroughly described.

Finally, to get dimensionless magnitudes, we measure the electron speed in units of its thermal velocity $v_0 = \sqrt{kT_0/m_e}$, where k is the Boltzmann constant, while we choose $t_0 = v_0^3/(n_e L^{e/e})$ as a time scale measure. Both parameters define $v_0 t_0$ as unit length whereas other scale factors for the thermodynamical forces, E and ∇T, can be derived from these three parameters.

3. Nonhomogeneous local Fokker–Planck equation

Up to this point we have not still dealt with the inhomogeneous terms that can be added to the collisional operator in velocity space. Once we define the precise short-time propagator in velocity space for advancing the homogeneous equation, the question of its use, when an effective source term is present, can be solved by means of (2), which specializes into the simple advance scheme given by

$$f_e(\mathbf{v}, t+\tau) = \int [f_e(\mathbf{v}', t) + \tau\rho(\mathbf{v}', t)] P_\tau(\mathbf{v}, \mathbf{v}')\, \mathrm{d}\mathbf{v}'. \tag{18}$$

In particular, the spatial inhomogeneities effects depicted by the Vlasov term have not been included in the above scheme. However, the inclusion of this term should pose no problem in dealing with an appropriate representation as a local approximation in configuration space. As a first approach to this problem, in [18] we formally replaced this contribution by an effective source term $\rho = \rho_{\rm spt}$ obtained from classical transport calculations followed by Spitzer and Harm [8]. In this sense, under the perspective of calculating the local transport coefficients, we obtained the steady state solution of (4), rewritten as

$$\frac{\partial f_e}{\partial t} = \rho + C(f_e, f_e), \tag{19}$$

with $\rho = \rho_{\rm spt}$ being a function of the generalized thermodynamical forces, $\nabla T = \partial T/\partial \mathbf{r}$ and $\mathbf{E}$, as

$$\rho_{\rm spt}(\nabla T, \mathbf{E}) = f_e^0 \left(\frac{mv^2}{2kT} - \frac{5}{2}\right) \frac{\mathbf{v}}{T} \cdot \frac{\partial T}{\partial \mathbf{r}} - f_e^0 \frac{e}{kT} \mathbf{E} \cdot \mathbf{v}. \tag{20}$$

Our solution was compared with that obtained in [8] from the equation $\rho_{\rm spt} = -C^{\rm lin}$ where $C^{\rm lin} = C\left(f_e^0, f_e^1\right)$ is the linearized collisional operator used in this classical calculations. As usual, in the linearization process the electron distribution function is assumed to have the form of a perturbed Maxwellian f_e^0 as $f_e = f_e^0(v) + f_e^1(\mathbf{v})$. The contributions of order $\left(f_e^1\right)^2$ in $C(f, f)$ are neglected even if a time advance or an iterative scheme is used to reach a steady state solution. We have obtained an excellent agreement in computing Spitzer's transport coefficients for vanishing small perturbative fields, these results being independent of any initial function $f_e(\mathbf{v}, 0)$. The difference between our equation and the Spitzer–Harm approximation lies in both using the full f_e, instead of its first-order linearized form, and substituting the linearized collisional term by the full nonlinear operator. In particular, from (20), we had $\mathbf{v} \cdot \nabla T = v_\parallel \partial T/\partial z$ and $\mathbf{v} \cdot \mathbf{E} = v_\parallel E$, where $v_\parallel$ means the velocity component in the direction of the electric field which is also assumed to be directed along the thermal gradient direction. Note that in the inhomogeneous term $\rho_{\rm spt}$, the temperature gradient factor is fixed in such a way that the average particle flux of the electrons vanishes in the absence of an electric field $\mathbf{E}$. Moreover, the electric field is included in its definition, this also assumes the collisional operator to be independent of both fields. Of course, the conditions under which transport is defined are, somewhat, arbitrary in the classical approach but our treatment with a nonlinearized operator suggested the possibility of replacing the so-called Vlasov term in the local approximation by $\rho_{\rm spt}$, although we had to adjust ourselves to similar physical conditions in order to obtain the same results. Nevertheless, physical inconsistencies were found in describing the distribution function that exhibited negative values for high v. These negative tails in f_e appear to be increasing in time, even for very small fields. Yet, it is interesting to recall that our nonlinearized collisional integral operator tends to correct these unphysical behaviour, appearing as a consequence of an ill-posed physical problem when $\rho_{\rm spt}$ is taken as a source term.

At a first approach in constructing a new source term, we investigate here the possibility of dealing with $\rho = \rho_{\rm spt}(\nabla T, 0)$, including the electric field strength into the direct propagator given in (10). Before we come to analyse our results in transport calculations, let us derive a set of fluid motion equations describing the role played by ρ in the time evolution of the distribution moments. To do this, we define the quantity $\langle g(\mathbf{v}, t)\rangle$ of any function g, depending explicitly on $\mathbf{v}$ and time t, as

$$\langle g\rangle(t) = \langle g(\mathbf{v}, t)\rangle = \int g(\mathbf{v}, t) f_e(\mathbf{v}, t)\, \mathrm{d}\mathbf{v} \tag{21}$$

that corresponds to the average of g in velocity space for a distribution f_e having $n_e = 1$. Taking into account a FPE in the form $\partial f/\partial t = \rho - \partial/\partial v_i[D_i - \partial D_{ij}/\partial v_j]f$, in the three-dimensional velocity space, the rate of change in time for $\langle g(\mathbf{v}, t)\rangle$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle g\rangle = \langle \dot{g}\rangle = \left\langle \frac{\partial g}{\partial t} + D_i \frac{\partial g}{\partial v_i} + D_{ij}\frac{\partial^2 g}{\partial v_i \partial v_j}\right\rangle + \int g\rho \,\mathrm{d}\mathbf{v} \tag{22}$$

for a well-behaved distribution function vanishing at infinite [2, 18]. Observe that, in our case, ρ may contain the Vlasov term, or any formal source term describing spatial inhomogeneities, and $D_i = D_i^e + F_i/m_e$. The values $g = 1$, $g = \mathbf{v}$ and $g = m_e v^2/2$ are clearly related to the number electron density n_e, bulk flow velocity $\mathbf{u} = \langle \mathbf{v}\rangle/n_e$, or electric current $\mathbf{J} = -en_e\mathbf{u}$, and the kinetic temperature $3kT/2 = \langle m_e v^2/2\rangle/n_e$. In the following, we assume that the rate of change for n_e, $\dot{n}_e = \int \rho d^3 v$, is zero under an appropriate definition of ρ. Therefore, the physically relevant quantities evolve in time throughout the motion equations, assuming a finite mass ratio μ,

$$\langle \dot{\mathbf{v}}\rangle = \left\langle \frac{\mathbf{F}}{m_e} + \mathbf{D}^{e/i}\right\rangle + \int \mathbf{v}\rho \,\mathrm{d}\mathbf{v} = \left\langle \frac{\mathbf{F}}{m_e}\right\rangle - \left(1 + \frac{\mu}{2}\right)\frac{n_i L^{e/i}}{4\pi}\left\langle \frac{\mathbf{v}}{v^3}\right\rangle + \int \mathbf{v}\rho \,\mathrm{d}\mathbf{v} \tag{23}$$

and

$$\frac{1}{2}\langle \dot{v^2}\rangle = \left\langle \mathbf{v}\cdot\frac{\mathbf{F}}{m_e}\right\rangle - \mu\frac{n_i L^{e/i}}{8\pi}\left\langle \frac{1}{v}\right\rangle + \frac{1}{2}\int v^2\rho \,\mathrm{d}\mathbf{v} \tag{24}$$

where we have taken into account that the electron–electron self-collision operator does not contribute to both momentum and energy exchanges. Observe that if f_e is not spatially homogeneous and we choose the Vlasov term as a possible source, $\rho = -(\partial/\partial\mathbf{r})\cdot\mathbf{v}f_e$, the above relations provide a set of differential fluid equations.

If $\rho = \rho_{\mathrm{spt}}(\nabla T, \mathbf{E})$ we have to take $\mathbf{F} = 0$ in the above fluid motion equations, since the external force $-eE$ is included in ρ. In this case, once a stationary steady state is reached, all the time derivatives vanish, so that the relations (for $\mu = 0$)

$$\int v_\parallel \rho \,\mathrm{d}\mathbf{v} = -\frac{eEn_e}{m_e} = \frac{n_e Z L^{e/e}}{4\pi}\left\langle \frac{v_\parallel}{v^3}\right\rangle \qquad \text{and} \qquad \frac{1}{2}\int v^2\rho \,\mathrm{d}\mathbf{v} = \mathbf{0} \tag{25}$$

hold for a large evolution time, as it actually happens. Here, use has been made of the relations $L^{e/i} = Z^2 L^{e/e}$ and $n_e = Zn_i$.

As previously remarked, we now test as a possible model for ρ the term $\rho = \rho_{\mathrm{spt}}(\nabla T, 0)$, as defined in (20), by adding the electric field to the drift vector in the Fokker–Planck operator and solving (19) with the scheme given in (18). As expected, similar inconsistencies as those found in [18] appear in this case. Besides the development of negative values in f_e during the time advance scheme, we have found that no steady state can be reached in the same cases. Furthermore, let us stress that from (23) and (24), in the stationary state

$$-\frac{eEn_e}{m_e} = \frac{n_e Z L^{e/e}}{4\pi}\left\langle \frac{v_\parallel}{v^3}\right\rangle \qquad \text{and} \qquad -\frac{eE}{m_e}\langle v_\parallel\rangle = 0 \tag{26}$$

should be satisfied (if $\mu = 0$), because $\int v^2\rho_{\mathrm{spt}}(\nabla T, 0)\,\mathrm{d}\mathbf{v} = 0$ and E has been added to the drift term of the collisional FPE operator. The last relation would give $\langle v_\parallel\rangle = n_e u_\parallel = 0$ for $t \to \infty$, which would seem to contradict the coherence of the classical scheme if no thermal gradient is adjusted to maintain zero current at any time t. This proves that $\rho = \rho_{\mathrm{spt}}(\nabla T, 0)$ would only be valid in the extreme limit of vanishing fields with the previous assumption about the existence of stationary solution for a linearized collisional operator, which is also assumed to be independent of the fields. It is important, anyway, to point out this discrepancy, because our results show a deviation of transport coefficients (in this case, electrical conductivity in

the absence of a thermal gradient) when the electric field starts to be high. In fact, exact coincidence with the Spitzer–Harm's result is only present for very weak fields, when the external force is properly included in the advance scheme as a part of the convective terms involved in the Fokker–Planck operator.

The previous analysis about using $\rho_{\mathrm{spt}}(\nabla T, 0)$ as a way to locally model the Vlasov term has shown the incoherence of such an election. Note also, that in the transient state (with $\dot{n}_e = 0$) we have $1/2\langle \dot{v}^2 \rangle = -eE/m_e\langle v_\parallel \rangle$, which explains the progressive increase of the system energy observed in our results for a finite electric field. For these reasons we are forced to seek for a new model for the term ρ.

4. A model for local transport coefficients calculus

The inconsistency of ρ_{spt} in describing local spatial inhomogeneities can be avoided by replacing it with a correct flux particle balance at a given point (small volume) in the space. Therefore, in this section we propose a new model of the inhomogeneous term ρ inspired in the previous interpretation of ρ_{spt}. This approach would also give rise to a method to deal with the Vlasov operator for local transport computation purposes. Our task now is to provide a model to compute transport coefficients, not only for very small fields and temperature gradients, but also for large departures from the equilibrium. Besides this task, we want to provide a time evolution for these coefficients, as well. At the same time, our aim is to describe a time-dependent physically coherent distribution function. For the sake of clarity, the transport coefficients are presented in units of the corresponding Spitzer–Harm results, and the same notation as used in [8] is preserved, that is

$$\mathbf{J} = \sigma \mathbf{E} + \alpha \nabla T, \qquad \mathbf{Q} = -\beta \mathbf{E} - \kappa \nabla T \tag{27}$$

for the electric current $\mathbf{J}$ and the total heat flux $\mathbf{Q}$, both computed in the reference system at rest with the ions.

Let us recall that a basic assumption in the classical theory is the existence of a local gradient of temperature and/or a local electric field at some point of space . Obviously, in fact, this is an idealization. Unless we are able to precisely describe how such gradients or fields are created, it makes little physical sense to postulate their existence. Our physical argument for constructing the source term is to prescribe the conditions under which either a temperature gradient or an electric field comes about. The way to achieve this is to define appropriate boundary conditions for the region of interest.

This procedure has been followed by other authors who studied the same problem in the past (see e.g. [30] and [31]). Once the gradients and fields exist, we must define our new source term ρ in such a way that it complies with the same physical conditions implicit in the analytical structure of ρ_{spt}. When this is achieved, it will be seen that our approach in transport calculations must coincide (except for small numerical errors) with those of the classical theory for very small gradients and fields. It is expected that only when the perturbations are significant do the results clearly deviate from the classical ones.

We will choose our usual reference frame in which the symmetry axis z in configuration space coincides with the $v_\parallel$ direction and, on the other hand, the thermal gradient and electric field are both directed along this same direction. Therefore, the space region in which transport is being analysed is a cylindrical slab of a certain width ΔZ, comprised between two planes taken as to be orthogonal to the symmetry axis. The existence of gradients, besides the forces in the region will be ensured if adequate boundary conditions are postulated on the left and right sides of the region of interest.

Such described geometry provides a straightforward expression for the $\mathbf{v}\cdot\partial f_e/\partial\mathbf{r}$ term in the equation for $\partial f_e/\partial t$, describing the density current change at the physical point of interest. This Vlasov term, as the divergence of a vector, can be understood as the flux of $\mathbf{v}f_e$ on the surface S of a small volume element $V(P)$ in configuration space, in the limit in which this volume reduces itself to a point P, $\nabla\cdot\mathbf{v}f_e = \lim_{V(p)\to 0}\int_S \mathbf{v}\cdot\mathrm{d}\mathbf{S}f_e/V(P)$. We choose to substitute such an idealized mathematical situation by a realistic small size slab region in contact with right (r) and left (l) boundary regions.

It is well known from other physical situations, as in electrostatics or fluid mechanics, that the corresponding boundary conditions may be regarded as a source distribution at the boundary surface enclosing a given volume. Clearly, in our slab region case, the $-\mathbf{v}\cdot\partial f_e/\partial\mathbf{r}$ term must be substituted by an effective local source term in the form

$$\rho = \rho(f_e,\mathbf{v}) = \frac{\Delta\Phi}{\Delta Z} = \frac{\Phi_l - \Phi_r}{\Delta Z}, \tag{28}$$

where Φ is a particle flux per unit area. Both fluxes $\Phi_{l,r}$, are defined as $\Phi_l = v_\parallel^+(f_l - f_e)$, $\Phi_r = v_\parallel^-(f_e - f_r)$ which corresponds to a particle balance through the left and right surfaces at $z = \mp\Delta Z/2$. The electron distribution functions on left and right sides are f_l and f_r, whereas $v_\parallel^\pm = (v_\parallel \pm |v_\parallel|)/2$. Obviously, we have assumed f_e anisotropized in the direction of the z axis, parallel to both driving forces $\mathbf{E}$ and ∇T, but a generalized ρ to any volume element, with a unit outward normal vector $\mathbf{n}$, can be obtained by a straightforward derivation. The influence of collisions and external forces on the system has already been evaluated through (17), besides (14) and (10). The time advance scheme is then provided by the simple relation (18) thanks to the short-time propagator P_τ adequate to the geometry implied in the definition of our formal source term ρ. In this sense, it must be stressed that our nonlinear collisional operator depends on the generalized thermodynamical forces at any time of the evolution, because the electron–electron drift and diffusion coefficients are computed at each time step.

For the purpose of obtaining the local transport coefficients, we must be sure that the same physical conditions are held here, as in the classical calculation. These conditions are reduced in practice to the requirement that the particle flux is not 're-normalized' in the perturbed system, as compared with the zero-order unperturbed Maxwellian.

In the dynamic solution of our equation (4), with the Vlasov term replaced by ρ, we must accordingly ensure that the final electron flux corresponding to the stationary function $f_e(\mathbf{v}, t\to\infty)$ also coincides with the initial electron flux. This is achieved if we advance the equation adjusting the densities at the left and right border regions at each time step, in such a way that both fluxes are equal and no density gradient is present. To measure transport, we have to maintain the temperatures of both boundaries constant, so that, the temperature gradient is fixed. We will, therefore, determine the left and right sides densities n_l, n_r from the simple flux balance conditions

$$\int \Phi_l\,\mathrm{d}\mathbf{v} = \int \Phi_r\,\mathrm{d}\mathbf{v} \qquad \text{and} \qquad \int v_\parallel\Phi_l\,\mathrm{d}\mathbf{v} = \int v_\parallel\Phi_r\,\mathrm{d}\mathbf{v} \tag{29}$$

assuming $f_{l,r}$ to be Maxwellian distribution functions with zero mean velocity and with temperatures $T_{l,r} = T \pm \Delta T/2$, as a first approach to test our model for ρ in this study. The central region is described by the function f_e at temperature T and density n_e. These conditions ensure electron density conservation in time, since they eliminate the contribution of ρ to the momentum rate of change (23), as it happens in Spitzer's calculations. However, the above relations do not drop the contribution of ρ in the energy transfer per unit of time in (24). Therefore, once a steady state is reached, we have that the time variation of the energy vanishes,

giving

$$\int \frac{1}{2}v^2\rho\,\mathrm{d}\mathbf{v} = \frac{eE}{m_e}\langle v_\parallel\rangle + \mu\frac{n_i L^{e/i}}{8\pi}\left\langle\frac{1}{v}\right\rangle \approx \frac{eE}{m_e}\langle v_\parallel\rangle. \tag{30}$$

This result eliminates the incoherence of (26) if $\rho_{\mathrm{spt}}(\nabla T, 0)$ is taken as a source of spatial inhomogeneities. If our source term is expanded up to first order in the temperature increment $\Delta T = T_r - T_l$, we find that $\Delta\Phi/\Delta Z$ and $\rho_{\mathrm{spt}}(\nabla T, 0)/|\nabla T|[(n_r T_r - n_l T_l)/(n_e \Delta Z)]$ only differ in a remainder of order $|\Delta T|^3$ in the first time step, when the evolution starts from Maxwellian distributions. This shows that for the first time step and small temperature gradients our convective term ρ approximates $\rho_{\mathrm{spt}}(\nabla T, 0)$, when the numerical parameters are appropriately determined. In our case, neglecting the small difference between the adjusted densities, we may consider the last expression in the bracket as our effective temperature gradient. Thus, if we measure the Spitzer gradient using as unit length the mean electron–ion collision length l_{ei}, we may write $\Delta Z = l_{\mathrm{ei}} N_C$, where N_C stands for the number of mean collision lengths corresponding to our slab width ΔZ. For the rest of the time steps, it is, of course, not possible to provide convenient analytical expressions as those above. In fact, our present convective term is not constant but evolving in time. At the steady state, however, it will probably differ little from the classical polynomial approximation. But its merit lies on the fact that it keeps the basic prescription of no flux renormalization, and does not induce unphysical negative amplitudes of f_e for large absolute values of $\mathbf{v}$. It should also be mentioned that the requirement of no density and flux renormalization does not force the system during its time evolution to have constant energy.

5. Evolution of transport coefficients and main results

As a first test of our effective source term added to the nonlinearized collisional FPE, we solve the problem of a Lorentz plasma (high Z limit) by eliminating the contribution of electron self-collisions $C^{e/e}$ in the advance scheme. In this test, we have found an almost exact coincidence with the transport coefficients given in [8] and other works for very weak fields. Therefore, for this kind of idealized plasma, no new results are found, in the sense that this system experiences the same qualitative behaviour we find for the more realistic case dealing with the electron collisional effects. Our results are summarized as follows.

In figure 1, the first row shows the time evolution of the current J (a) and heat flux Q (b) in the particular case in which an external field is introduced, having the same dimensionless size as the temperature gradient. The resulting current and heat flux are normalized at any time t since they are divided by the corresponding current and heat flux in Spitzer's calculation for the identical cases. In these plots, we see that lower fields than the dimensionless $E = -|\nabla T| = 10^{-5} = -\mathrm{grad}\,T$ are necessary for an exact coincidence with Spitzer's results. For a real plasma with $n_e = 10^{20}$ m^{-3} at $T = 1$ KeV, a dimensionless field of order $E \sim 2\times 10^{-3}$ corresponds to a field of order 1 V m^{-1} . As in [30] and [31], we find that our solution approaches the classical results when a large number of mean collision lengths ($N_C \gtrsim 10^2$) are used in defining the spatial scale ΔZ. In these cases, for very small fields and gradients, the asymptotic values approaching unity corresponds respectively to the values $(\sigma - \alpha)$ and $(\beta - \kappa)$ computed with Spitzer's distribution function f_{spt}.

It can be seen that for increasing fields, both fluxes, J and Q, are reduced up to a 20% with respect to the ideal 'zero fields' limit. The role played by the high velocity electron population is substantive in these results. Observe that our f_e remain positive at any time (frames f and g), while f_{spt} (frame h) develops considerably large negative amplitudes in the distribution tails. These negative tails increase as the temperature gradient scale becomes comparable to

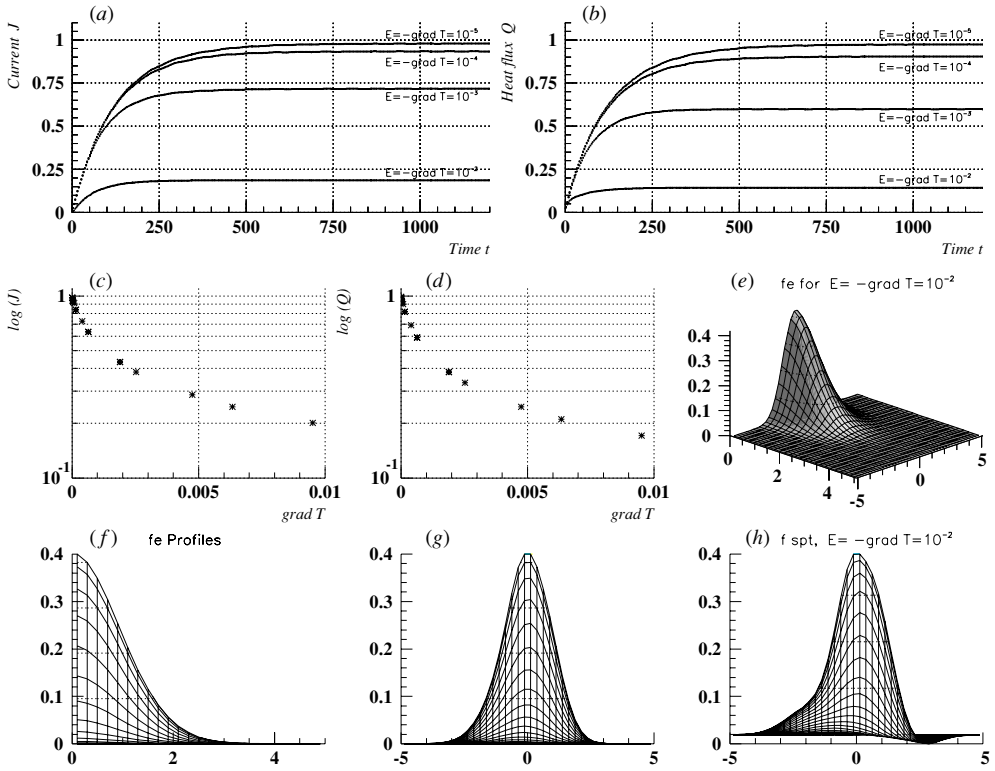

Figure 1. Time evolution of the electric current J and the heat flux Q for different values of dimensionless temperature gradient *grad* T and electric field (*a*), (*b*). The dependence of both fluxes on *grad* T is shown in (*c*) and (*d*). The electron distribution function profiles in the steady state are plot in (*e*), (*f*) and (*g*), while (*h*) shows the contour of the Spitzer's distribution having negative amplitudes.

the collision length of the electrons having velocities of order $5v_0/2$, or higher. Our f_e is also anisotropic in the direction of the z axis, remaining close to the Maxwellian distributions $f_{l,r}$ at the boundaries of our slab region, but differing substantially from a Maxwellian function when ∇T and E are large. Similar departures from the classical transport coefficients are found if $f_{l,r}$ are taken as anisotropic distributions. The nonlinear relation between fluxes and ∇T is clearly depicted by the graphs plotted in the second row of the figure. The first two frames (c, d) describe the dependence on the current and heat flux (in logarithmic scale) with the temperature gradient, in absence of electric fields, giving α and κ. Note that the unit values (Spitzer's coefficients) are approached for extremely small gradients.

Finally, to end this study, let us underline the generality of our procedure. We have adjusted our calculation to the precise physical conditions leading to the definition of the classical transport coefficients. Yet, a wealth of physical problems requiring very different conditions in their time evolution can also be addressed, both for their transient and final stationary solutions. With this model the extraordinary complexity introduced in transport computation by a magnetic field can be simplified by a suitable redefinition of our term ρ. In this case, the flux particle balance must be evaluated on the surfaces of a small space volume, with each outward normal unit vectors in the parallel and orthogonal directions to the field. The resulting scalar term $\rho(f_e, \mathbf{v})$ can be manageable under change of variables in velocity

space, if needed. The efficiency and robustness of our integral collisional operator makes it quite realistic to study the interaction of several space regions, in which the distribution functions are defined and external boundary conditions close the system. These topics will be the object of a future research.

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